COMBINATORICA 5 (3) (1985) 193—204

HOW MANY ATOMS CAN BE DEFINED BY BOXES?

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Received 3 May 1983

We study the function $b(n, d)$, the maximal number of atoms defined by n d-dimensional boxes, i.e. parallelopipeds in the d-dimensional Euclidean space with sides parallel to the coordinate axes.

We characterize extremal interval families defining $b(n, 1)=2n-1$ atoms and we show that $b(n, 2)=2n^2-6n+7$.

We prove that for every d,

$b^*(d)=\lim_{n\to\infty} b(n, d)/n^d$ exists and $1 \leqq (d/2)\sqrt[d]{b^*(d)} \leqq e$.

Moreover, we obtain $b^*(3)=8/9$.

1. Introduction

Let $\mathscr{A}_n=\{A_1, A_2, \ldots, A_n\}$ be a family of non-empty sets. The elements $x, y \in \bigcup_{i=1}^{n} A_i$ are said to be equivalent if for all i, $1 \leqq i \leqq n$, $x \in A_i$ if and only if $y \in A_i$. The equivalence classes are called the *atoms* of $\mathscr{A}_n$.

R. Rado studied the number of atoms for families given by immersion in a Euclidean space ([5]). In particular, he raised the problem of determining the maximal number of atoms in a family of *n d-dimensional boxes,* i.e. n parallelopipeds of the d-dimensional Euclidean space $\mathbf{R}^d$, with sides parallel to the coordinate axes. This number is denoted by $b(n, d)$ and families defining $b(n, d)$ atoms are called *extremal.* R. Rado proved in [5] that $b(n, 1)=2n-1$ and gave upper bounds for $b(n, d)$ and a lower bound for $b(n, 2)$.

The problem of determining the number (or the maximal number) of atoms for a family of figures of the Euclidean space is apparently new. However, some of our results (e.g. Remark 4.6.) show that this problem closely relates to a more familiar area of combinatorial geometry: to find the number (or maximal number) of *connected regions* defined by a family of figures. An excellent review is given by B. Grünbaum in [3].

AMS subject classification (1980): 51 M 05, 52 A 20.

This paper presents results on $b(n, d)$. We use the well-known notion of the *overlap graph* (cf. [2]) to study the 1-dimensional case, i.e. the number of atoms defined by families of *intervals*. We show that the number of atoms defined by n intervals is $2n-c$ where c denotes the number of connected components in the overlap graph (Theorem 2.1). As a consequence, we get a characterization of extremal interval families. An interval family is extremal if and only if its overlap graph is connected (Corollary 2.2).

Rectangles or 2-dimensional boxes are discussed in section 3. We introduce the notion of *overlay index* and we show its role related to the number of atoms in rectangle families (Theorems 3.1 and 3.2). These investigations lead to the main result of section 3: $b(n, 2)=2n^2-6n+7$ (Theorem 3.5). We note that the extremal families have an interesting structure, their complete description is given in [4].

We do not know the exact value of $b(n, d)$ for $d\geqq 3$. In section 4 we show the existence of

$$b^*(d) = \lim_{n\to\infty} b(n, d)/n^d$$

for every fixed d (Theorem 4.3). We prove that $b^*(3)=8/9$ (Corollary 4.7). For $d\geqq 4$ we show that for a suitable constant $c\geqq 1$, $c\leqq (d/2)\sqrt[d]{b^*(d)}\leqq e$ (Corollary 4.4). (The best value of c, known for us, is $\sqrt[3]{3}$ if d is a multiple of 3.)

Throughout the paper we impose two restrictions on the family of boxes, having no effect on the maximal number of atoms. We assume that the boxes are *closed*. Applying small enlargements for the boxes of a family, the number of atoms does not decrease, therefore we always assume that the *boundary hyperplanes of the boxes are all different.*

We note that extremal families of intervals may contain disjoint intervals. However, for higher dimensions the situation changes: *every extremal system must have non-empty intersection if* $d\geqq 2$ (Lemma 3.3).

2. Intervals

Let $\mathscr{I}=\{I_1, I_2, \ldots, I_n\}$ be a family of intervals (all closed and having no common endpoints). Let $a(\mathscr{I})$ denote the number of atoms defined by $\mathscr{I}$. The *overlap graph* of $\mathscr{I}$, $G(\mathscr{I})$, is defined on the vertex set $\{1, 2, \ldots, n\}$ and uv is an edge of $G(\mathscr{I})$ if and only if I_u and I_v overlap, i.e., they intersect but neither properly contains the other. As far as we know, overlap graphs were used first by Fulkerson and Gross [1] in the study of interval families; for further results see [2].

Theorem 2.1. *If the overlap graph of an interval family $\mathscr{I}$ consists of c connected components then $\mathscr{I}$ defines* $2|\mathscr{I}|-c$ atoms.

Proof. Let $G_1, G_2, \ldots, G_c$ be the connected components of $G(\mathscr{I})$ and consider the subfamilies

$$\mathscr{I}_j = \{I_v\in\mathscr{I}: v \text{ is a vertex in } G_j\} \quad (1\leqq j\leqq c).$$

Replace in $\mathscr{I}$ the members of $\mathscr{I}_1$ by the interval $U=\bigcup_{I\in\mathscr{I}_1} I$: $\mathscr{I}'=(\mathscr{I}\setminus\mathscr{I}_1)\cup\{U\}$. It is easy to see that U overlaps no interval of $\mathscr{I}'$ and thus $a(\mathscr{I})=a(\mathscr{I}_1)+a(\mathscr{I}')$.

The repeated application of this argument yields $a(\mathscr{I})=\sum_{j=1}^{c} a(\mathscr{I}_j)$, consequently it is sufficient to show that if $\mathscr{I}$ has a connected overlap graph then $a(\mathscr{I})=2|\mathscr{I}|-1$.

The set $\bigcup_{I\in\mathscr{I}} I$ is an interval splitted by $2|\mathscr{I}|-2$ endpoints into $2|\mathscr{I}|-1$ open intervals. We will prove that points of different intervals belong to different atoms by showing that every atom defined by $\mathscr{I}$ is convex. Indeed, let p and q be two arbitrary points of an atom A and denote by P the interval with endpoints p and q. Since P overlaps no interval of $\mathscr{I}$, the overlap graph of the subfamily $\mathscr{J}=\{I\in\mathscr{I}: I\subset P\}$ and the overlap graph of the subfamily $\mathscr{K}=\{I\in\mathscr{I}: I\supset P\}$ belong to different components of $G(\mathscr{I})$. By the connectedness of $G(\mathscr{I})$ it follows that $\mathscr{K}\neq\emptyset$ and $\mathscr{J}=\emptyset$, consequently $P\subset A$.

Corollary 2.2. *If $\mathscr{I}$ is a family of intervals then*
(i) $|\mathscr{I}|\leqq a(\mathscr{I})\leqq 2|\mathscr{I}|-1$;
(ii) *$a(\mathscr{I})$ can have every value between $|\mathscr{I}|$ and $2|\mathscr{I}|-1$;*
(iii) *$a(\mathscr{I})=2|\mathscr{I}|-1$ if and only if $G(\mathscr{I})$ is connected.* ∎

Two extremal interval families are displayed on Fig. 1.

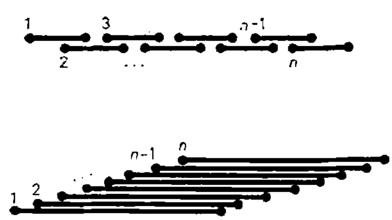

Fig. 1. Two extremal interval families with overlap graphs P_n and K_n

3. Rectangles

In this section we deal with families of 2-dimensional boxes, i.e. rectangles in the plane, and we determine $b(n, 2)$.

We introduce the notion of overlay index of families of rectangles as follows.

Let $\mathscr{B}$ be a family of rectangles and v_1, v_2, v_3 and v_4 be the four vertices of some rectangle $R\in\mathscr{B}$. If the vertex v_i is contained by n_i rectangles different from R then $\omega(v_i)=\max\{0, n_i-1\}$ is called the overlay index of v_i and

$$\omega(R) = \sum_{i=1}^{4} \omega(v_i)$$

is called the *overlay index* of R. The overlay index of the whole family is defined as $\omega(\mathscr{B})=\sum_{R\in\mathscr{B}} \omega(R)$ (cf. Fig. 2.).

Theorem 3.1. *If $\mathscr{B}$ is a family of n rectangles with non-empty intersection then $\mathscr{B}$ defines at most $2n^2-5n+5-\omega(\mathscr{B})$ atoms.*

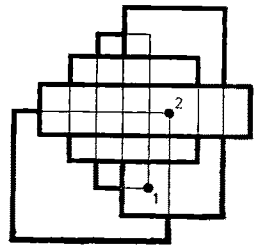

Fig. 2. Five rectangles with total overlay index 3

Proof. Suppose that the interior of the set $\bigcap_{B\in\mathscr{B}} B$ contains the coordinate origin. The boundaries of the rectangles split the plane into connected domains. Let $\mathscr{D}$ be the set of domains which are covered by *at least two* rectangles of $\mathscr{B}$. Denote by $\mathscr{D}_1$ the set of those domains from $\mathscr{D}$ which meet the positive orthant.

Every domain $D\in\mathscr{D}_1$ has an upper right corner which is either the intersection of the boundary lines of two rectangles $P, Q\in\mathscr{B}$ or it is the vertex of some $R\in\mathscr{B}$ and lies in some $S\in\mathscr{B}$. To get an upper bound for $|\mathscr{D}_1|$, we associate to $\mathscr{D}$ the pair P and Q, or the pair R and S. In the second case there are $\omega_1(R)+1$ possibilities to choose S, where $\omega_1(R)$ denotes the overlay index of R at its vertex belonging to the positive orthant. Therefore $|\mathscr{D}_1|\leqq\binom{n}{2}-\sum_{R\in\mathscr{B}}\omega_1(R)$ and since this argument is true for the other three orthants and for the corresponding sets $\mathscr{D}_2$, $\mathscr{D}_3$ and $\mathscr{D}_4$, we get

$$(1)\qquad \sum_{i=1}^{4}|\mathscr{D}_i| \leqq 4\binom{n}{2}-\sum_{i=1}^{4}\sum_{R\in\mathscr{B}}\omega_i(R) = 2n^2-2n-\omega(\mathscr{B}).$$

By taking into account the domains which meet several orthants, we give an upper bound for the cardinality of $\mathscr{D}$.

For $j=2, 3$ and 4 let $\mathscr{D}^{(j)}\subset\mathscr{D}$ be the set of domains which meet j orthants and intersect the coordinate cross $x\cup y$ in $j-1$ intervals. Denote by $\mathscr{D}^{(0)}$ the set consisting of the domain which contains the origin and r further twofold connected domains which meet $x\cup y$ in four intervals $(0\leqq r\leqq n-2)$. Each coordinate axis is splitted by the boundary segments of the domains of $\mathscr{D}$ into $2n-3$ intervals, thus

$$|\mathscr{D}^{(2)}|+2|\mathscr{D}^{(3)}|+3|\mathscr{D}^{(4)}|+4|\mathscr{D}^{(0)}|-2 = 4n-6,$$

and clearly $|\mathscr{D}^{(0)}|=r+1$. From here and by inequality (1)

$$|\mathscr{D}| = \sum_{i=1}^{4}|\mathscr{D}_i|-(|\mathscr{D}^{(2)}|+2|\mathscr{D}^{(3)}|+3|\mathscr{D}^{(4)}|+3|\mathscr{D}^{(0)}|) \leqq$$

$$\leqq 2n^2-2n-\omega(\mathscr{B})-(4n-5-r),$$

that is

$$(2)\qquad |\mathscr{D}| \leqq 2n^2-6n+5-\omega(\mathscr{B})+r.$$

The atoms of $\mathscr{B}$ not appearing in $\mathscr{D}$ are those covered by just one rectangle. Obviously the number of these atoms is at most $n-r$ since the largest twofold connected domain of $\mathscr{D}^{(0)}$ surrounds at least r rectangles. Thus, using (2), we obtain: $a(\mathscr{B})\leqq|\mathscr{D}|+n-r\leqq 2n^2-5n+5-\omega(\mathscr{B})$. ∎

The projections of a rectangle family $\mathscr{B}$ into the coordinate axes x and y give two interval families. Their overlap graphs, denoted by $G_h(\mathscr{B})$ and $G_v(\mathscr{B})$, are called the *horizontal* and *vertical overlap graphs* of $\mathscr{B}$.

Theorem 3.2. *Let $\mathscr{B}$ be a family of rectangles with non-empty intersection. If the vertical and the horizontal overlap graphs of the rectangles are connected then the overlay index of $\mathscr{B}$ is at least $|\mathscr{B}|-2$.*

Proof. Let $\mathscr{B}=\{B_1, B_2, ..., B_n\}$ and $n\geqq 3$. By the connectivity of the vertical overlap graph G_v, it can be assumed that the top-line of B_1 and the bottom-line of B_2 meet every rectangle B_i, $1\leqq i\leqq n$. Moreover, one can suppose that the rectangles are indexed in such a way that for every j, $3\leqq j\leqq n$,

(a) the top-line of B_j or the bottom-line of B_j meets every rectangle B_k with $j<k\leqq n$, and

(b) for some index $i<j$, B_i and B_j are vertically overlapping: ij is an edge of G_v.

Indeed, supposing that the first $j-1$ rectangles of $\mathscr{B}$ satisfy (a) and (b), the j'th rectangle can be chosen as follows. There is a rectangle from $\mathscr{B}\setminus\{B_1, ..., B_{j-1}\}$ which vertically overlaps some B_i, $1\leqq i\leqq j-1$, since G_v is connected. If the top-line of B_i meets every rectangle from $\mathscr{B}\setminus\{B_1, ..., B_i\}$ then let B_j be the rectangle whose bottom-line intersects every rectangle of $\mathscr{B}\setminus\{B_1, ..., B_{j-1}\}$; otherwise, B_j will be the rectangle whose top-line meets every member of $\mathscr{B}\setminus\{B_1, ..., B_{j-1}\}$.

Observe that among the eight vertices of B_1 and B_2 exactly two, say s and t belong to $B_1\cap B_2$. Hence, every rectangle B_i, satisfying $3\leqq i\leqq n$ and $\{s,t\}\cap B_i\neq\emptyset$, increases the overlay index of $\{B_1, B_2\}$, i.e. if $K=\{i\colon 3\leqq i\leqq n,\ \{s,t\}\cap B_i=\emptyset\}$ then $\omega(B_1)+\omega(B_2)\geqq n-2-|K|$. Thus, it is sufficient to show that

$$\sum_{3\leqq i\leqq n}\omega(B_i)\geqq|K|.$$

Let us consider a spanning tree T_h of the connected horizontal overlap graph and direct the edges of T_h in such a way that every vertex different from 1 have outdegree one. Denote by j^+ the endpoint of the edge starting from vertex j, $2\leqq j\leqq n$.

By property (b), for every $j\in K$ there exists a rectangle $B_{j'}$, $j'<j$ and $j'<j^+$, such that $B_{j'}$ vertically overlaps one of B_j and B_{j^+}. Using property (a), it is easy to verify that there is a vertex p among the twelve vertices of $B_{j'}$, B_j and B_{j^+} which is

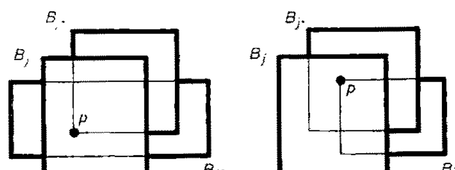

Fig. 3. The triplet (j', j, j^+) defines overlay index increment at vertex p

covered twice by these rectangles (see the two essentially different situations on Fig. 3.). If this p is a vertex of B_k ($k=j'$, j or j^+) then we say that the triplet (j', j, j^+) belongs to B_k and overlays p. Remark that no triplet belongs to B_1 or B_2, since if $j\in K$ then B_j does not overlay the vertices of B_1 and B_2.

Let p be the vertex of some rectangle B_k, $3\leqq k\leqq n$, and denote by $E(p)$ the set of ordered pairs jj^+ defined by the triplets (j', j, j^+) belonging to B_k and overlaying p. Since ordered pairs defined by different triplets are different,

$$\sum_p |E(p)| \geqq |\{(j', j, j^+)\colon j\in K\}| = |K|. \tag{3}$$

If $E(p)\neq\emptyset$ for some vertex p of a given B_k then $E(p)$ is the edge set of a subforest of the tree T_h; denote by $V(p)$ its vertex set. For the minimal element j_0 (or j_0^+) of $V(p)$, clearly $j_0'\notin V(p)$ holds, therefore $\omega(p)\geqq\left|(V(p)\cup\{j_0'\})\setminus\{k\}\right|-1=$ $=|V(p)|-1\geqq|E(p)|$. From here, by (3), $\sum_{3\leqq k\leqq n}\omega(B_k)\geqq\sum_p|E(p)|\geqq|K|$ follows. ∎

Lemma 3.3. *Every extremal family of d-dimensional boxes has non-empty intersection if* $d\geqq 2$.

Proof. Let $\mathscr{B}$ be an extremal family of n boxes in $\mathbf{R}^d$ and suppose that there are disjoint intervals on some coordinate axis among the projections of the boxes of $\mathscr{B}$. We define an operation on $\mathscr{B}$ which reduces the number of non-intersecting pairs of intervals on the coordinate axes in such a way that the number of box atoms does not decrease.

Consider the projection of $\mathscr{B}$ on the i'th coordinate axis and choose two disjoint intervals $[p_1, p_2]$ (the projection of $P\in\mathscr{B}$) and $[q_1, q_2]$ (the projection of $Q\in\mathscr{B}$) such that $p_2<q_1$, moreover, no interval endpoints lie in the open segment (p_2, q_1).

Change P by P' and Q by Q', replacing $[p_1, p_2]$ by $[p_1, q_1]$ and $[q_1, q_2]$ by $[p_2, q_2]$. Then every atom defined by $\mathscr{B}$ which has a representative point outside of the strip $S=\{(x_1, \dots, x_d)\in\mathbf{R}^d\colon p_2\leqq x_i\leqq q_1\}$ remains atom for the modified family $\mathscr{B}'$. If an atom A defined by $\mathscr{B}$ is inside of S, i.e. it has no representative point outside of S, then for every $(a_1, \dots, a_d)\in A$, $(a_1, \dots, a_{i-1}, p_2, a_{i+1}, \dots, a_d)\in P$ and $(a_1, \dots, a_{i-1}, q_1, a_{i+1}, \dots, a_d)\in Q$. Thus different atoms inside the strip S remain different for $\mathscr{B}'$ and since the set $P'\cap Q'\subset S$ contains them, they are all different from the atoms represented outside of S. Therefore $a(\mathscr{B}')\geqq a(\mathscr{B})$, moreover if $P'\cap Q'\neq\emptyset$ and S is at least 2-dimensional then the atoms of $\mathscr{B}$ inside of S can not cover $P'\cap Q'$ therefore $a(\mathscr{B}')>a(\mathscr{B})$ (cf. Fig. 4.).

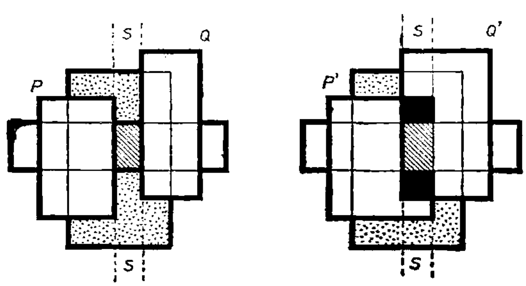

Fig. 4. The boxes P and Q become intersecting and give birth to the black atom

After repeated applications of the above operation $P' \cap Q' \neq \emptyset$ occures which would increase $a(\mathscr{B})$, contradicting the extremal property of $\mathscr{B}$. ∎

Lemma 3.4. *For every n there exists an n-element extremal family of rectangles with connected vertical and horizontal overlap graphs.*

Proof. Let $\mathscr{B}$ be an extremal family of n rectangles. By Lemma 3.3, $\bigcap_{B \in \mathscr{B}} B \neq \emptyset$. Suppose that one of the overlap graphs of $\mathscr{B}$, say the vertical overlap graph G_v is not connected. This means that the top-line e of $P \in \mathscr{B}$ and a bottom-line f of $Q \in \mathscr{B}$ separate the rectangles of $\mathscr{B}$, i.e. $\{B \in \mathscr{B}\colon B \cap e \neq \emptyset\} = \{B \in \mathscr{B}\colon B \cap f \neq \emptyset\} \neq \mathscr{B}$.

To prove the lemma we show a transformation of $\mathscr{B}$ which reduces the number of the connected components of G_v in such a way that the number of atoms does not decrease.

Let g be the closest top-line to e among the top-lines under e and suppose that g belongs to $R \in \mathscr{B}$. If we modify P and R exchanging their top-lines e and g then they become vertically overlapping (cf. Fig. 5.). On the other hand, the number of atoms does not decrease, since every atom meeting the open strip between e and g before the transformation has a representative point on the bottom-line f. ∎

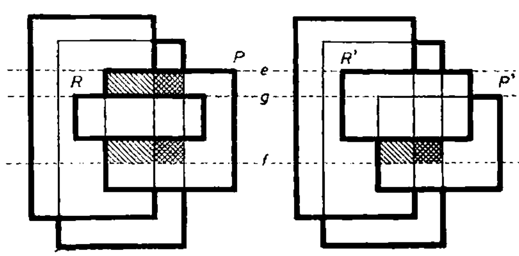

Fig. 5. Exchange of two top-lines

Theorem 3.5. *For every* $n \geqq 2$, $b(n, 2) = 2n^2 - 6n + 7$.

Proof. Let $\mathscr{B}$ be an extremal family of n rectangles. By Lemmas 3.3 and 3.4 one can suppose that the rectangles have a common point and both overlap graphs are connected. By Theorem 3.2, $\omega(\mathscr{B}) \geqq n-2$ and from Theorem 3.1, $b(n, 2) = a(\mathscr{B}) \leqq \leqq 2n^2 - 5n + 5 - \omega(\mathscr{B}) \leqq 2n^2 - 6n + 7$.

The next construction shows that $b(n, 2) \geqq 2n^2 - 6n + 7$ for every $n \geqq 2$. We give the rectangles R_k $(1 \leqq k \leqq n)$ as direct products of closed intervals in a coordinate system of the plane:

$$R_k = [-k, k] \times [-n+1+k, n-1-k] \quad \text{if} \quad 1 \leqq k \leqq n-2,$$

$$R_{n-1} = [-n+1, n-1] \times [-n, 0] \quad \text{and}$$

$$R_n = [-n, 0] \times [-n+1, n-1].$$

The family $\mathscr{R}=\{R_1, R_2, \ldots, R_n\}$ splits the plane into $4\binom{n-1}{2}$ unit squares and 3 further finite connected domains (cf. Fig. 6.). All of these domains represent different atoms and therefore

$$b(n, 2) \geqq a(\mathscr{R}) = 4\binom{n-1}{2}+3 = 2n^2-6n+7. \qquad \blacksquare$$

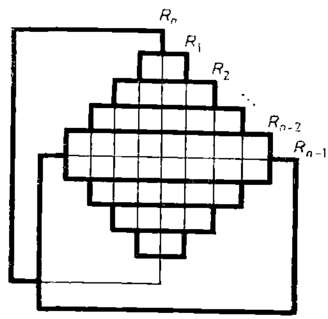

Fig. 6. An extremal rectangle family

4. General estimations

When examining $b(n, d)$, it is useful to consider set systems consisting of so called corners. A *d-dimensional corner* at $\mathbf{v}=(v_1, \ldots, v_d)\in\mathbf{R}^d$ is the set of points $\mathbf{x}=(x_1, \ldots, x_d)$ such that $x_i \leqq v_i$ whenever $1\leqq i\leqq d$; in other words, corners are the translates of the negative orthant of $\mathbf{R}^d$.

We define $c(n, d)$ as the maximal number of atoms in a family consisting of n d-dimensional corners. We assume, just as in the case of boxes, that the corners have no common boundary hyperplanes. The close connection between $b(n, d)$ and $c(n, d)$ will be pointed out in Theorem 4.2 saying that their ratio is approximately 2^d.

First we give estimations on $c(n, d)$. It is straightforward that $c(n, 1)=n$ and $c(n, 2)=n+\binom{n}{2}=\binom{n+1}{2}$.

The key notion of this section is the *representation of atoms by index sets.* Consider a family of corners C_i at $\mathbf{v}_i=(v_1^i, \ldots, v_d^i)$. Let A be an atom of this family and $I=\{i: A\subset C_i\}$. For $1\leqq k\leqq d$, choose $i_k\in I$ such that C_{i_k} is minimal in the k'th coordinate, i.e. $v_k^{i_k}<v_k^j$ if $i_k\neq j\in I$. We say that $I(A)=\{i_k: 1\leqq k\leqq d\}$ *represents A*. Of course, $|I(A)|\leqq d$ and different atoms are represented by different index sets.

Theorem 4.1. *For every fixed* $d\geqq 3$,

$$\left(\frac{n}{d}\right)^d - O(n^{d-1}) \leqq c(n, d) \leqq \frac{2}{d}\binom{n}{d}+O(n^{d-1}).$$

Proof. *1. Upper bound.* Let us consider a family of corners C_i at $\mathbf{v}_i=(v_1^i, v_2^i, \ldots, v_d^i)$, $i=1, \ldots, n$. The number of atoms A with $|I(A)|<d$ is at most

$$(4) \qquad \binom{n}{1}+\ldots+\binom{n}{d-1} = O(n^{d-1}).$$

Let $c_0(n, d)$ denote the maximal number of atoms A with $|I(A)|=d$ in families of n d-dimensional corners and assume that $c_0(n, d)$ is achieved by our family $C_1, \ldots, C_n$. According to (4),

$$(5) \qquad c(n, d) \leqq c_0(n, d)+\sum_{i=1}^{d-1}\binom{n}{i}.$$

We assume that the corners are indexed in such a way that for every k, $1 \leqq k \leqq d$, the corner vertices satisfy $\min_{k \leqq i \leqq n} v_k^i=v_k^k$. Any $(d-1)$-element subfamily of $\{C_{d+1}, \ldots, C_n\}$ has a member C_m such that, in this subfamily, $\mathbf{v}^m$ is minimal in the j_1'th and the j_2'th coordinates $(1 \leqq j_1<j_2 \leqq d)$. Let I be the index set of this subfamily completed by some index k, $1 \leqq k \leqq d$, different from j_1 and j_2. Then I does not represent any atom, since one of $\mathbf{v}^m$ and $\mathbf{v}^k$ gives the minimum in two different coordinates. The d-element subsets with the properties above can be chosen in $(d-2)\binom{n-d}{d-1}$ different ways. Repeating this procedure $\lfloor n/d \rfloor$ times for the subfamilies $\{C_{d+1}, \ldots, C_n\}$, $\{C_{2d+1}, \ldots, C_n\}$, and so on, we obtain

$$(6) \qquad c_0(n, d) \leqq \binom{n}{d}-(d-2) \sum_{i=1}^{\lfloor n/d \rfloor}\binom{n-id}{d-1}.$$

Since $(d-2) \sum_{i=1}^{\lfloor n/d \rfloor}\binom{n-id}{d-1}=\frac{d-2}{d}\binom{n}{d}-O(n^{d-1})$, (4), (5) and (6) imply $c(n, d) \leqq \frac{2}{d}\binom{n}{d}+O(n^{d-1})$.

2. Lower bound. We define a family of n corners at vertices $\mathbf{v}^1, \ldots, \mathbf{v}^n$ in the following way:

$$v_j^i= \begin{cases}i & \text{if } \; i \equiv j \pmod d. \\ 2n-i & \text{otherwise}\end{cases}$$

Every d-element set $I \subset\{1, \ldots, n\}$ containing pairwise non-congruent indices mod d represents some atom for the family, since, for every $k \in I$, $\mathbf{v}^k$ is minimal in the j'th coordinate where $k \equiv j \pmod d$, $1 \leqq j \leqq d$. Consequently, $c(n, d) \geqq(n/d)^d$ for every n divisible by d.

For every n, $(n/d)^d-\lfloor n/d\rfloor^d=O(n^{d-1})$, therefore $c(n, d) \geqq(n/d)^d-O(n^{d-1})$ follows. ∎

Theorem 4.2. *For every* $d \geqq 1$, $\lim_{n \to \infty} \frac{b(n, d)}{c(n, d)}=2^d$.

Proof. We proceed in 3 steps, proving:

(a) $b(n, d) \leqq 2^d c(n, d)$

(b) $b(n, d) \geqq 2^d c(n-d, d)$

(c) $\lim\limits_{n\to\infty} \dfrac{c(n-d, d)}{c(n, d)} = 1.$

(a) Let the boxes $B_1, \dots, B_n$ form a family having $b(n, d)$ atoms. By Lemma 3.3, we suppose that the origin is inside $B_1 \cap \dots \cap B_n$. Then, in an arbitrary orthant of $\mathbf{R}^d$, the boxes can be considered as corners, therefore in a fixed orthant they can define at most $c(n, d)$ atoms. As the number of orthants is exactly 2^d, $b(n, d) \leqq 2^d c(n, d)$ follows.

(b) Consider an extremal family of $n-d$ corners in $\mathbf{R}^d$, and suppose that the origin is contained by every corner. Let C_i be the corner at $\mathbf{v}^i = (v_1^i, \dots, v_d^i)$, $d+1 \leqq i \leqq n$. Then the points $\{(y_1, \dots, y_d)\colon |y_1| = v_1^i, \dots, |y_d| = v_d^i\}$ can be taken as the vertices of a box B_i. Let $m > \max\{v_j^i\colon 1 \leqq j \leqq d,\ d+1 \leqq i \leqq n\}$ and for every k, $1 \leqq k \leqq d$, $B_k = \{(x_1, \dots, x_d)\colon -m \leqq x_i \leqq m,\ x_k \leqq 0\}$. The definition of $B_1, \dots, B_d$ ensures that the $2^d c(n-d, d)$ atoms defined by $B_{d+1}, \dots, B_n$ in the orthants are different.

(c) Every index $i \leqq n$ is contained by at most $\binom{n-1}{d-1} + o(n^{d-1})$ representing sets $I(A) \subset \{1, \dots, n\}$. Therefore, deleting d arbitrary corners from an extremal family with $c(n, d)$ atoms, we have

$$c(n, d) \leqq c(n-d, d) + d\binom{n-1}{d-1} + o(n^{d-1}) = c(n-d,\ d) + o(n^d).$$

By Theorem 4.1, $o(n^d)$ can be replaced by $o\big(c(n, d)\big)$ and the proof is done because, on the other hand, $c(n, d) \geqq c(n-d, d)$ is trivial. ∎

Theorem 4.3. *For every* $d \geqq 1$, $b^*(d) = \lim\limits_{n\to\infty} \dfrac{b(n, d)}{n^d}$ *exists.*

Proof. By Theorem 4.2, it is enough to prove that

$$c^*(d) = \lim_{n\to\infty} \frac{c(n, d)}{n^d}$$

exists. To achieve this, we show the existence of

$$\lim_{n\to\infty} \frac{\binom{n}{d} - c_0(n, d)}{n^d}; \tag{7}$$

with the $c_0(n, d)$ defined in the proof of Theorem 4.1 as the maximal number of atoms represented by d-element index sets in a family of n d-dimensional corners. From (5) $c(n, d) - c_0(n, d) = o(n^d)$.

The proof is based on the following general observation. Let $\mathcal{H}$ be a class of d-uniform hypergraphs without multiple edges. Suppose that $\mathcal{H}$ is closed under the operation of vertex deletion and, for every n, there exists an $H \in \mathcal{H}$ on n vertices. Define $f(n) = \min \{|E(H)|: H \in \mathcal{H}, |V(H)| = n\}$, where $E(H)$ and $V(H)$ denote the set of edges and vertices of H respectively. Then $\lim_{n \to \infty} \frac{f(n)}{n^d}$ exists.

We apply this observation for the following class $\mathcal{H}$ of hypergraphs. For every family of d-dimensional corners $C_1, \ldots, C_n$, a d-uniform hypergraph H is defined: $V(H) = \{1, \ldots, n\}$ and a d-element set I is an edge of H if and only if I does not represent an atom in the family. Now clearly $f(n) = \binom{n}{d} - c_0(n, d)$, therefore the existence of (7) follows. ∎

Corollary 4.4. *For every* $d \geqq 1$, $1 \leqq \frac{d}{2} \sqrt[d]{b^*(d)} \leqq e$. ∎

The methods presented here lead to sharp results in the 3-dimensional case.

Theorem 4.5. *For every* $n \geqq 1$, $c(n, 3) = \left\lfloor \frac{(n+1)^3+1}{9} \right\rfloor$.

Proof. For $d = 3$, (5) and (6) imply that

$$c(n, 3) \leqq n + \binom{n}{2} + \binom{n}{3} - \sum_{i=1}^{\lfloor n/3 \rfloor} \binom{n-3i}{2}. \tag{8}$$

Let $\mathcal{C}$ be the family of n corners given in the proof for the lower bound of Theorem 4.1. Since no corner contains any other corner of $\mathcal{C}$, every one- and two-element index set represents an atom.

Let $\{k, l, m\}$ be a three-element index set with $1 \leqq k < l < m \leqq n$. Obviously, this set does not represent an atom of $\mathcal{C}$ if and only if $k \equiv l \pmod 3$. The number of these sets with $3(i-1)+1 \leqq k \leqq 3i$ is $\binom{n-3i}{2}$ for every i, $1 \leqq i \leqq \lfloor n/3 \rfloor$. Therefore

$$c(n, 3) \geqq n + \binom{n}{2} + \binom{n}{3} - \sum_{i=1}^{\lfloor n/3 \rfloor} \binom{n-3i}{2}. \tag{9}$$

The explicit form of the right hand side of (9) is $\left\lfloor \frac{(n+1)^3+1}{9} \right\rfloor$, thus by (8) and (9) the theorem follows. ∎

Remark 4.6. The atoms defined by families of corners are connected regions therefore $c(n, d)$ gives the maximal number of connected regions defined by n corners. Theorem 4.5, for example can be stated as follows: n translates of an orthant in $\mathbf{R}^3$ divides $\mathbf{R}^3$ into at most $\left\lfloor \frac{(n+1)^3+1}{9} \right\rfloor + 1$ connected regions.

Corollary 4.7. $b^*(3) = 8/9$. ∎

Conjecture 4.8. *For* $n \geqq 3$, $b(n, 3) = 8 \left\lfloor \frac{(n-2)^3+1}{9} \right\rfloor + 7$.

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